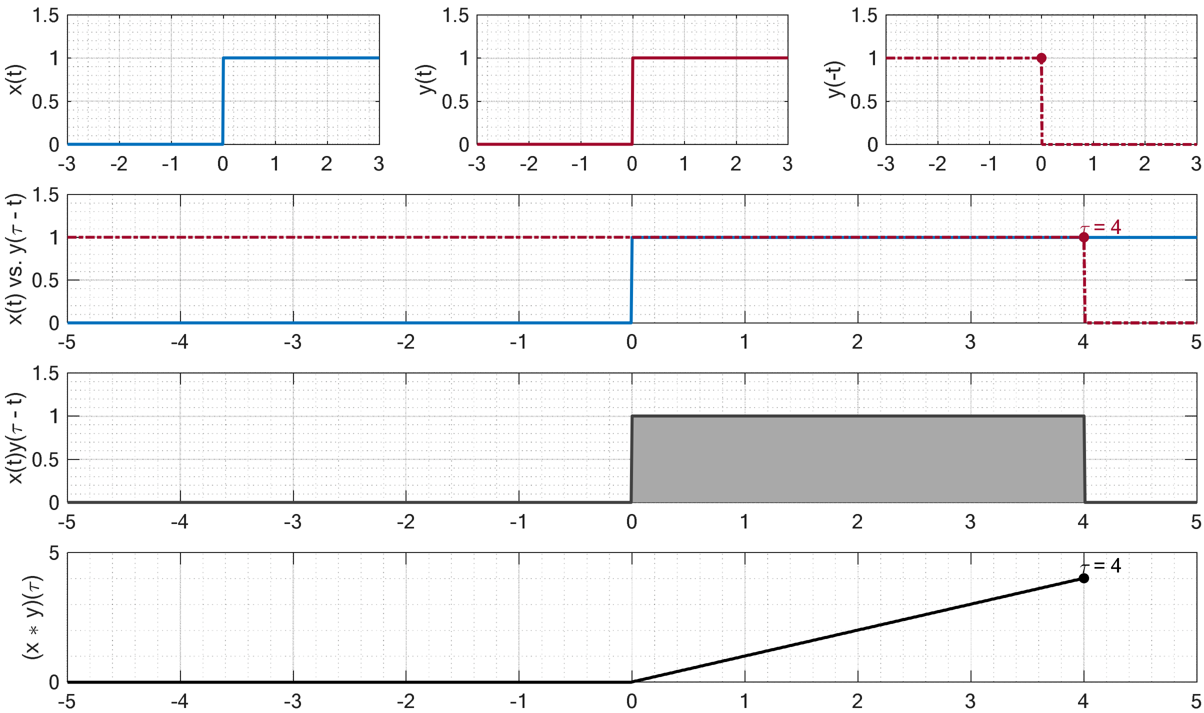
Convolution: Practice Problems

Q1. For the signals $x(t) = u(t)$ and $y(t) = u(t)$, determine the convolution result $x(t) * y(t)$.

Q1. Solution.



Region #1: For time-shift $\tau < 0$

$$z(\tau) = x(t) * y(t) = \int_{-\infty}^{\infty} x(t)\, y(\tau - t)\, dt$$

$$z(\tau) = \int_{-\infty}^{\infty} (0)\, dt = 0$$

Region #2: For time-shift $\tau \geq 0$

$$z(\tau) = x(t) * y(t) = \int_{-\infty}^{\infty} x(t)\, y(\tau - t)\, dt$$

$$z(\tau) = \int_{0}^{\tau} 1\, dt = [t]_0^{\tau} = \tau$$

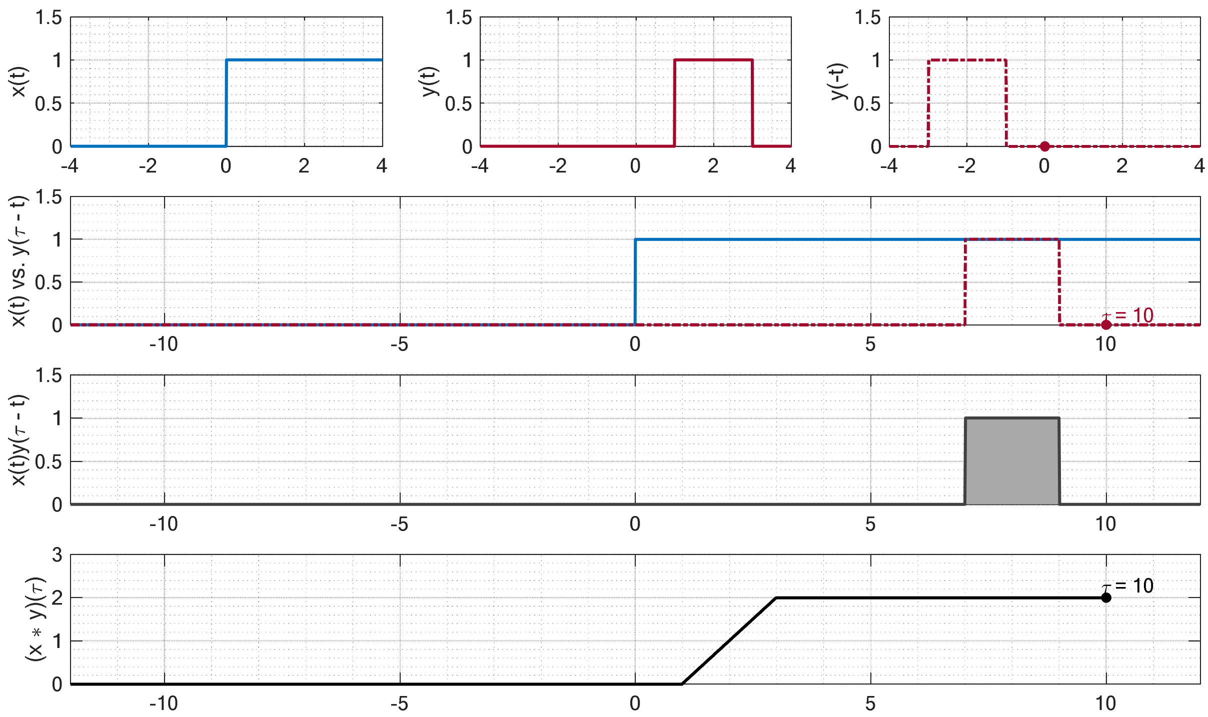
Full Solution (the ramp signal):

$$z(\tau) = x(t) * y(t) = \text{ramp}(\tau) = \tau\, u(\tau) = \begin{cases} 0, & \tau < 0 \\ \tau, & \tau \geq 0 \end{cases}$$

For an animation of the graphical solution, please watch the YouTube video (https://www.youtube.com/watch?v=GeJ7UAb2vVk).

Q2. For the signals $x(t) = u(t)$ and $y(t) = \text{rect}\left(\frac{t-2}{2}\right)$, determine the convolution result $x(t) * y(t)$.

Q2. Solution. To perform the graphical solution, first draw $x(t)$, $y(t)$ and then $y(-t)$ as follows



Region #1: For time-shift $\tau < 1$

$$z(\tau) = x(t) * y(t) = \int_{-\infty}^{\infty} x(t)\, y(\tau - t)\, dt$$

$$z(\tau) = \int_{-\infty}^{\infty} (0)\, dt = 0$$

Region #2: For time-shift $1 \leq \tau < 3$

$$z(\tau) = x(t) * y(t) = \int_{-\infty}^{\infty} x(t)\, y(\tau - t)\, dt$$

$$z(\tau) = \int_{0}^{\tau-1} 1 \times 1\, dt = 1 \times 1 \times [t]_{0}^{\tau-1} = \tau - 1$$

Region #3: For time-shift $\tau \geq 3$

$$z(\tau) = x(t) * y(t) = \int_{-\infty}^{\infty} x(t)\, y(\tau - t)\, dt$$

$$z(\tau) = \int_{\tau-3}^{\tau-1} (1 \times 1)\, dt = 1 \times 1 \times [t]_{\tau-3}^{\tau-1} = 2$$

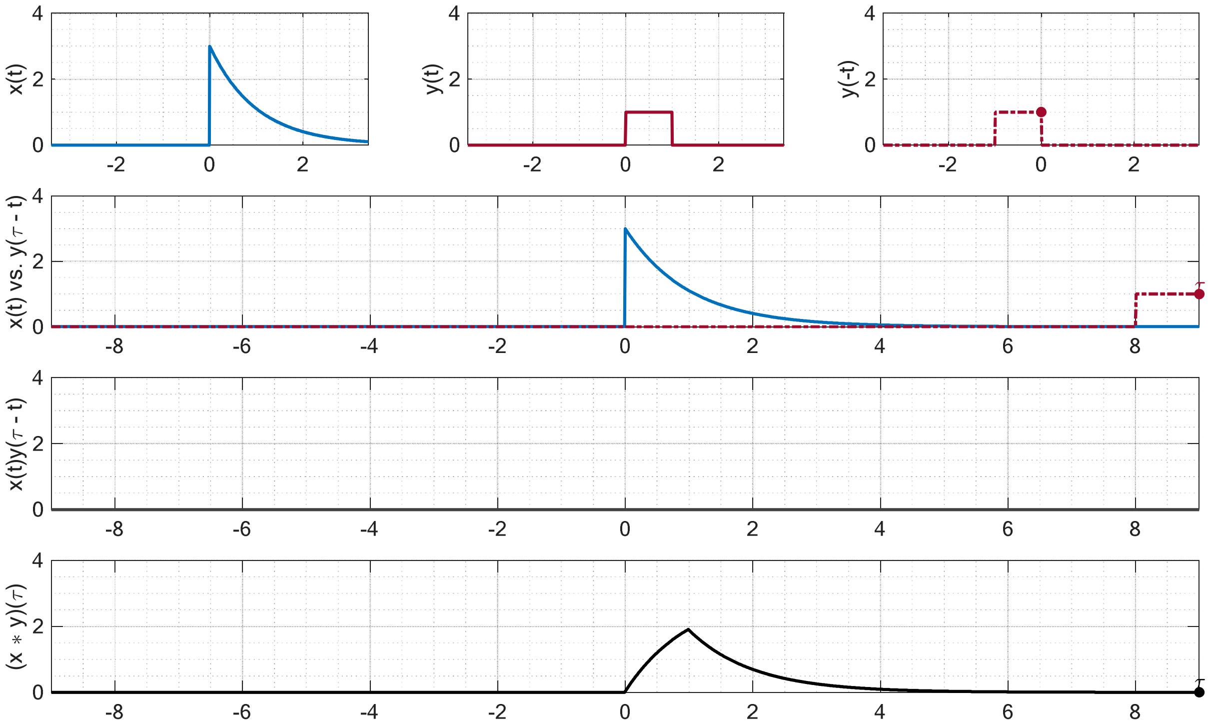
Full Solution (three regions):

$$z(\tau) = x(t) * y(t) = \begin{cases} 0, & \tau < 1 \\ \tau - 1, & 1 \leq \tau < 3 \\ 2, & \tau \geq 3 \end{cases}$$

For an animation of the graphical solution, please watch the YouTube video (https://www.youtube.com/watch?v=GeJ7UAb2vVk).

Q3. For the signals $x(t) = 3e^{-t}u(t)$ and $y(t) = \text{rect}(t - 0.5)$, determine the convolution result $x(t) * y(t)$.

Q3. Solution. To perform the graphical solution, first draw $x(t)$, $y(t)$ and then $y(-t)$ as follows



Region #1: For time-shift $\tau < 0$

$$z(\tau) = x(t) * y(t) = \int_{-\infty}^{\infty} x(t)\, y(\tau - t)\, dt$$

$$z(\tau) = \int_{-\infty}^{\infty} (0)\, dt = 0$$

Region #2: For time-shift $0 \le \tau < 1$

$$z(\tau) = x(t) * y(t) = \int_{-\infty}^{\infty} x(t)\, y(\tau - t)\, dt$$

$$z(\tau) = \int_{0}^{\tau} 3e^{-t} \times 1\, dt = 3 \times \left[\frac{e^{-t}}{-1}\right]_0^{\tau} = 3(1 - e^{-\tau})$$

Region #3: For time-shift $\tau \ge 1$

$$z(\tau) = x(t) * y(t) = \int_{-\infty}^{\infty} x(t)\, y(\tau - t)\, dt$$

$$z(\tau) = \int_{\tau-1}^{\tau} 3e^{-t} \times 1\, dt = 3 \times \left[\frac{e^{-t}}{-1}\right]_{\tau-1}^{\tau} = 3e^{-\tau}(e^{1} - 1)$$

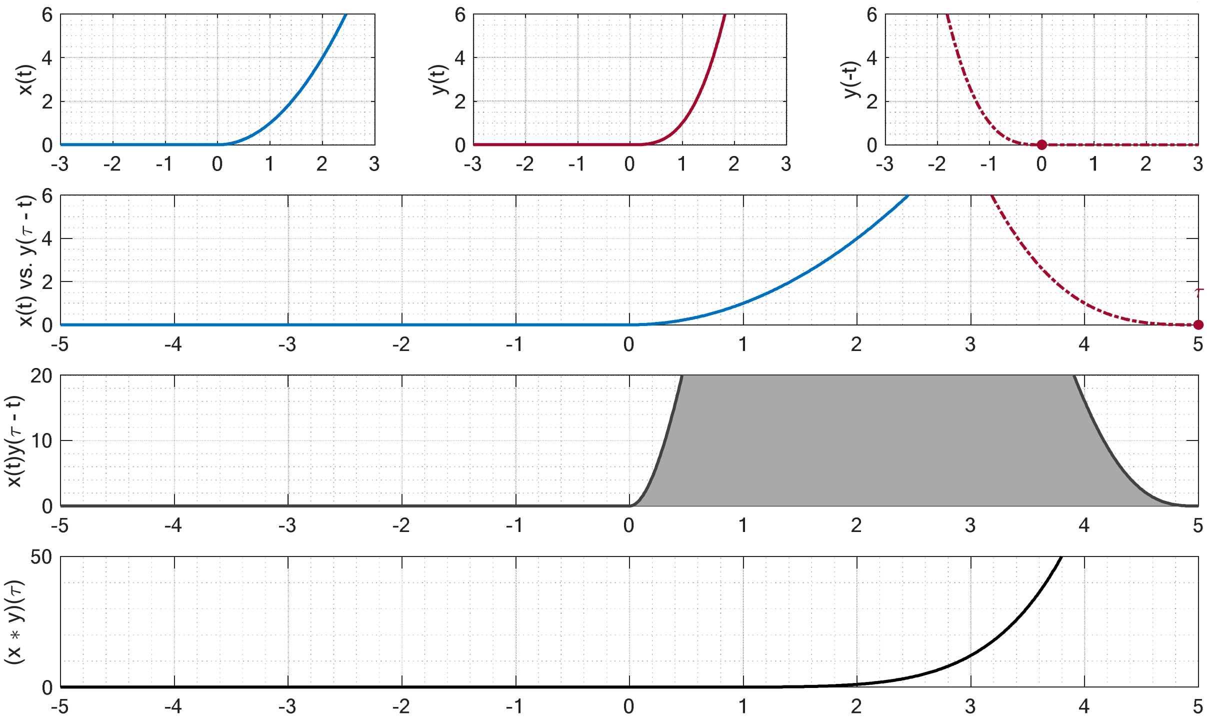
Full Solution (three regions):

$$z(\tau) = x(t) * y(t) = \begin{cases} 0, & \tau < 0 \\ 3(1 - e^{-\tau}), & 0 \leq \tau < 1 \\ 3e^{-\tau}(e^{1} - 1), & \tau \geq 1 \end{cases}$$

For an animation of the graphical solution, please watch the YouTube video (https://www.youtube.com/watch?v=GeJ7UAb2vVk).

Q4. For the signals $x(t) = t^3 u(t)$ and $y(t) = t^2 u(t)$, determine the convolution result $x(t) * y(t)$.

Q4. Solution. To perform the graphical solution, first draw $x(t)$, $y(t)$ and then $y(-t)$ as follows



Region #1: For time-shift $\tau < 0$

$$z(\tau) = x(t) * y(t) = \int_{-\infty}^{\infty} x(t)\, y(\tau - t)\, dt$$

$$z(\tau) = \int_{-\infty}^{\infty} (0)\, dt = 0$$

Region #2: For time-shift $\tau \geq 0$

$$z(\tau) = x(t) * y(t) = \int_{-\infty}^{\infty} x(t)\, y(\tau - t)\, dt$$

$$z(\tau) = \int_{0}^{\tau} t^3 \times (\tau - t)^2\, dt = \left[\frac{\tau^2\, t^4}{4} + \frac{-2\tau\, t^5}{5} + \frac{t^6}{6}\right]_0^{\tau} = \frac{\tau^6}{60}$$

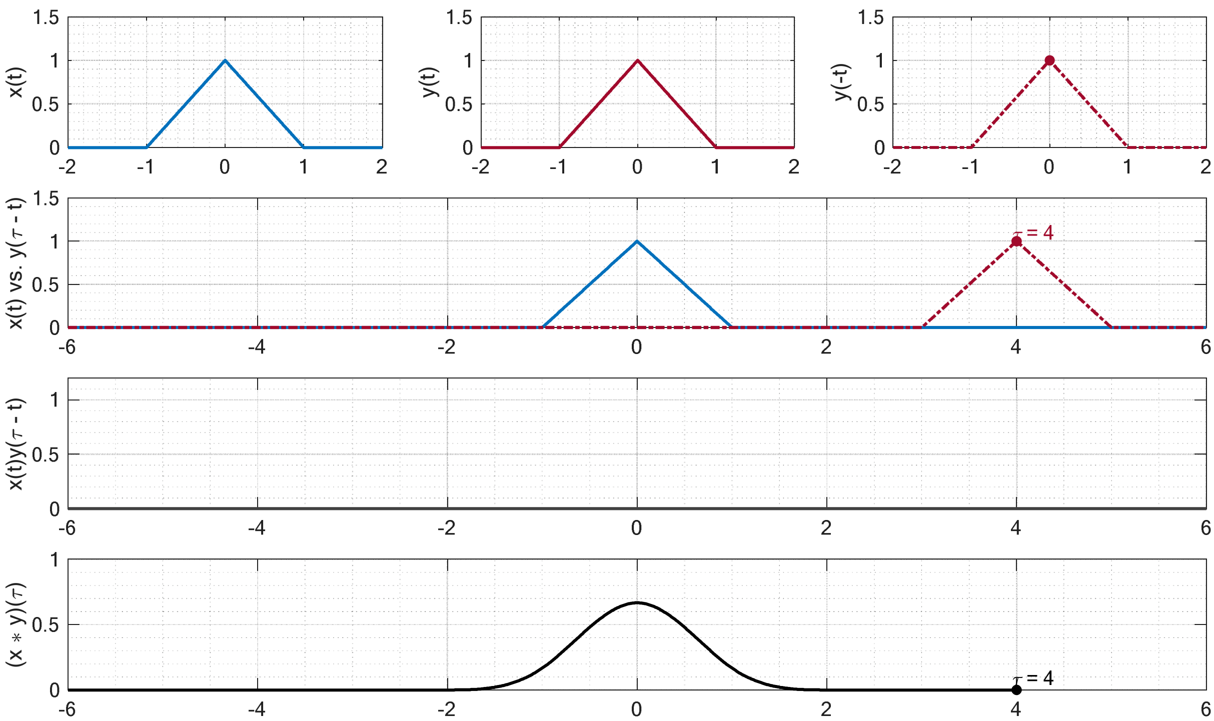
Full Solution (two regions):

$$z(\tau) = x(t) * y(t) = \frac{\tau^6}{60} u(t) = \begin{cases} 0, & \tau < 0 \\ \dfrac{\tau^6}{60}, & \tau \geq 0 \end{cases}$$

For an animation of the graphical solution, please watch the YouTube video (https://www.youtube.com/watch?v=GeJ7UAb2vVk).

Q5. For the signals $x(t) = \Delta(t)$ and $y(t) = \Delta(t)$, determine the convolution result $x(t) * y(t)$.

Q5. Solution. To perform the graphical solution, first draw $x(t)$, $y(t)$ and then $y(-t)$ as follows



Region #1: For time-shift $\tau < -2$

$$z(\tau) = x(t) * y(t) = \int_{-\infty}^{\infty} x(t)\, y(\tau - t)\, dt$$

$$z(\tau) = \int_{-\infty}^{\infty} (0)\, dt = 0$$

Region #2: For time-shift $-2 \leq \tau < -1$

$$z(\tau) = x(t) * y(t) = \int_{-\infty}^{\infty} x(t)\, y(\tau - t)\, dt$$

$$z(\tau) = \int_{-1}^{\tau+1} (1 + t)(1 - t + \tau)\, dt = \frac{1}{6}(\tau + 2)^3$$

Region #3: For time-shift $-1 \leq \tau < 0$

$$z(\tau) = x(t) * y(t) = \int_{-\infty}^{\infty} x(t)\, y(\tau - t)\, dt$$

$$z(\tau) = \int_{-1}^{\tau} (1 + t)(1 + t - \tau)\, dt + \int_{\tau}^{0} (1 + t)(1 - t + \tau)\, dt$$
$$+ \int_{0}^{\tau+1} (1 - t)(1 - t + \tau)\, dt$$
$$= \frac{1}{3}(-\tau^3 + 3\tau + 2) - \frac{1}{6}\tau(\tau^2 + 6\tau + 6)$$

Region #4: For time-shift $0 \leq \tau < 1$

$$z(\tau) = x(t) * y(t) = \int_{-\infty}^{\infty} x(t)\, y(\tau - t)\, dt$$

$$z(\tau) = \int_{\tau-1}^{0} (1 + t)(1 + t - \tau)\, dt + \int_{0}^{\tau} (1 - t)(1 + t - \tau)\, dt$$
$$+ \int_{\tau}^{1} (1 - t)(1 - t + \tau)\, dt$$
$$= \frac{1}{3}(\tau^3 - 3\tau + 2) + \frac{1}{6}\tau(\tau^2 - 6\tau + 6)$$

Region #5: For time-shift $1 \leq \tau < 2$

$$z(\tau) = x(t) * y(t) = \int_{-\infty}^{\infty} x(t)\, y(\tau - t)\, dt$$

$$z(\tau) = \int_{\tau-1}^{1} (1 - t)(1 + t - \tau)\, dt = \frac{-1}{6}(\tau - 2)^3$$

Region #6: For time-shift $\tau \geq 2$

$$z(\tau) = x(t) * y(t) = \int_{-\infty}^{\infty} x(t)\, y(\tau - t)\, dt$$

$$z(\tau) = \int_{-\infty}^{\infty} (0)\, dt = 0$$

Full Solution (two regions):

$$z(\tau) = x(t) * y(t) = \begin{cases} 0, & \tau < -2 \\ \frac{1}{6}(\tau + 2)^3, & -2 \leq \tau < -1 \\ \frac{1}{3}(-\tau^3 + 3\tau + 2) - \frac{1}{6}\tau(\tau^2 + 6\tau + 6), & -1 \leq \tau < 0 \\ \frac{1}{3}(\tau^3 - 3\tau + 2) + \frac{1}{6}\tau(\tau^2 - 6\tau + 6), & 0 \leq \tau < 1 \\ \frac{-1}{6}(\tau - 2)^3, & 1 \leq \tau < 2 \\ 0, & \tau \geq 2 \end{cases}$$

For an animation of the graphical solution, please watch the YouTube video (https://www.youtube.com/watch?v=GeJ7UAb2vVk).