Signal Analysis Student Guide

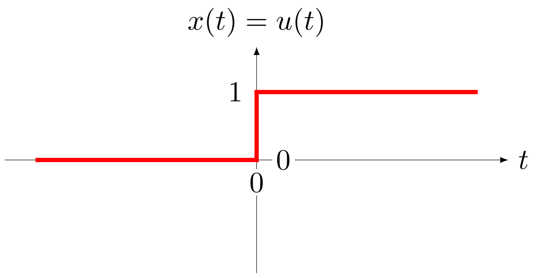
Prepared by: Prof. Mohammed Hawa

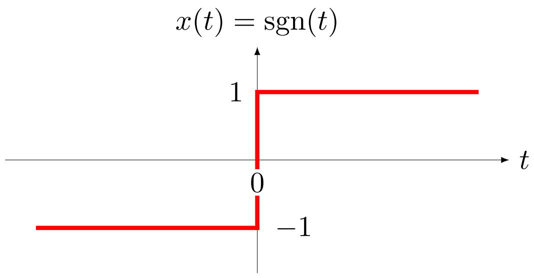
October 2, 2024

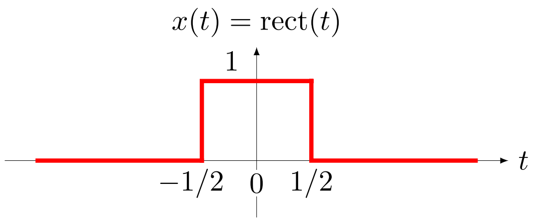
This handout provides you with a quick review of the main concepts and equations you studied in signal analysis, such as Fourier series, Fourier transform, average value, and average power.

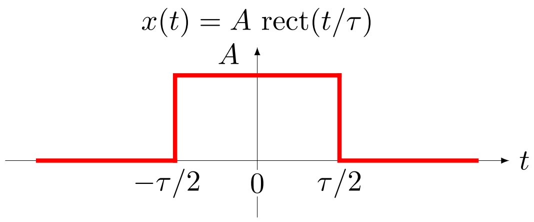
1 Basic Signals

The following figures show some basic signals you need to know, along with their notation. For example, $x(t) = A\ \mathrm{rect}(t/\tau)$ is a single rectangular pulse of *width* τ, while $x(t) = A\ \Delta(t/\tau)$ is a single triangular pulse of width 2τ. All these signals are *aperiodic* signals. However, we can build *periodic* signals by repeating the aperiodic pulse shape every *period* of time T (the *period* T should not to be confused with the *pulse width* τ). An example is shown later in Exercise 1, where we repeat the rectangular pulse to obtain a corresponding periodic signal. The $\mathrm{rep}_T\{.\}$ operator is used to indicate repetition every T, hence the signal in Exercise 1 is expressed as $x(t) = \mathrm{rep}_T\{A\ \mathrm{rect}(t/\tau)\}$.

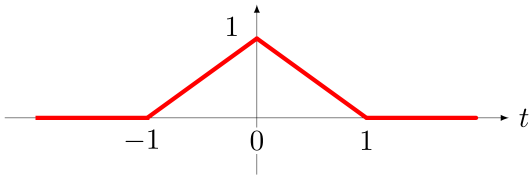




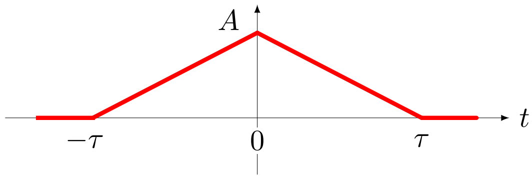




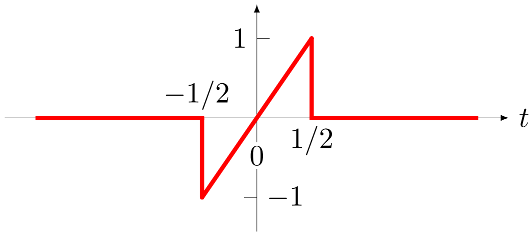
$x(t) = \Delta(t)$



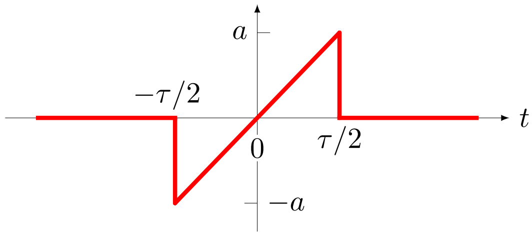
$x(t) = A\ \Delta(t/\tau)$



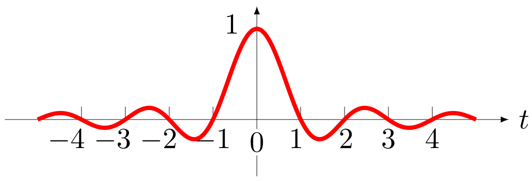
$x(t) = \mathrm{saw}\ (t)$



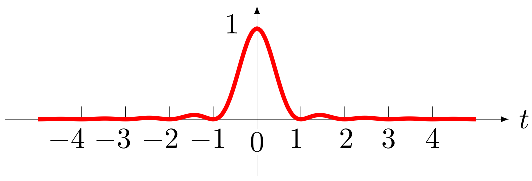
$x(t) = a\ \mathrm{saw}\ (t/\tau)$



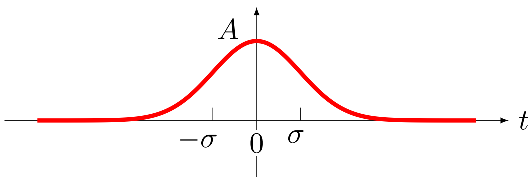
$x(t) = \mathrm{sinc}(t) = \sin(\pi t)/\pi t$



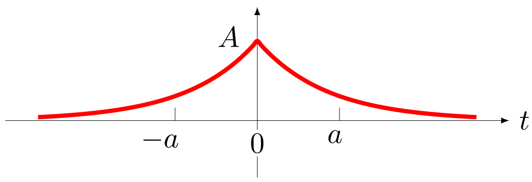
$x(t) = \mathrm{sinc}^2(t)$



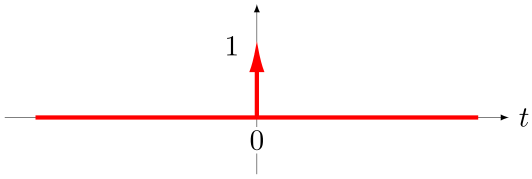
$x(t) = A\ e^{-t^2/\left(2\sigma^2\right)}$



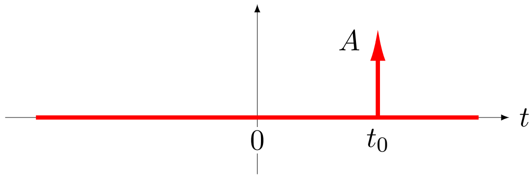
$x(t) = A\ e^{-a|t|}$



$x(t) = \delta(t)$



$x(t) = A\ \delta(t - t_0)$



2 Fourier Series

The idea behind Fourier series is that you can express any *periodic* signal $x(t)$ as the sum of an infinite number of sinusoidal (cosine/sine) functions. There are three ways of doing that: complex exponential form, trigonometric form and compact form. The complex exponential form is the one used heavily in this course.

2.1 Complex exponential Fourier series

Any *periodic* signal $x(t)$ with period T can be expanded into complex *exponential* Fourier series as follows:

$$x(t) = \sum_{n=-\infty}^{\infty} \alpha_n \, e^{jn\omega_0 t} \; , \qquad \omega_0 = \frac{2\pi}{T}$$

where,

$$\alpha_n = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) e^{-jn\omega_0 t} dt, \quad n = 0, \ \pm 1, \ \pm 2, \ \pm 3, \ \ldots$$

For real-valued $x(t)$ we have $\alpha_{-n} = \alpha_n^*$ (i.e., $|\alpha_{-n}| = |\alpha_n|$, $\angle\alpha_{-n} = -\angle\alpha_n$), $n = 0, \ 1, \ 2, \ 3, \ \ldots$

2.2 Trigonometric Fourier series

Any *periodic* signal $x(t)$ with period T can be expanded into *trigonometric* Fourier series as:

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \right] \quad , \qquad \omega_0 = \frac{2\pi}{T}$$

where,

$$a_n = 2 \; Re\{\alpha_n\} = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \cos(n\omega_0 t) \, dt \quad , \qquad n = 0, \ 1, \ 2, \ 3, \ldots$$

$$b_n = -2 \; Im\{\alpha_n\} = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \sin(n\omega_0 t) \, dt \quad , \qquad n = 0, \ 1, \ 2, \ 3, \ldots$$

Notice that,

$$\alpha_n = \frac{a_n}{2} - j\frac{b_n}{2} \quad , \qquad n = 0, \ 1, \ 2, \ 3, \ldots$$

2.3 Compact Fourier series

Any *periodic* signal $x(t)$ with period T can be expanded into *compact* Fourier series as:

$$x(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t - \theta_n), \quad \omega_0 = \frac{2\pi}{T}$$

where,

$$c_n = \sqrt{a_n^2 + b_n^2} = 2\,|\alpha_n| \;\; , \qquad n = 0,\ 1,\ 2,\ 3, \ldots$$

$$\theta_n = \tan^{-1}\left(\frac{b_n}{a_n}\right) = \tan^{-1}\left(\frac{-Im\left\{\alpha_n\right\}}{Re\left\{\alpha_n\right\}}\right) = -\angle\alpha_n, \quad n = 0,\ 1,\ 2,\ 3,\ \ldots$$

Notice that,

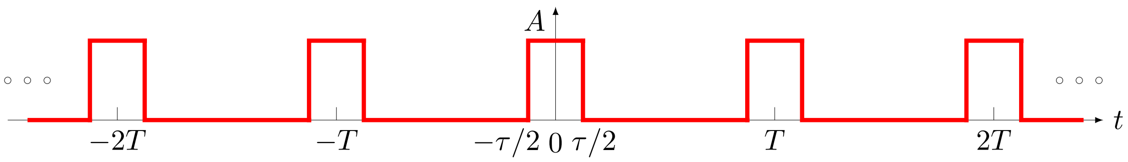
$$a_n = c_n \cos\left(\theta_n\right) = 2\ Re\left\{\alpha_n\right\}, \quad n = 0,\ 1,\ 2,\ 3,\ \ldots$$

$$b_n = c_n \sin\left(\theta_n\right) = -2\ Im\left\{\alpha_n\right\}, \quad n = 0,\ 1,\ 2,\ 3,\ \ldots$$

Not only do you need to know the above equations, you also need to be able to use them to evaluate the Fourier series coefficients for any periodic function $x(t)$. To practice your skills, the exercises below show some periodic signals $x(t)$ with the corresponding Fourier series coefficients. Make sure you can obtain them yourself.

Exercise 1. For the signal $x(t)$ shown below, evaluate and sketch the Fourier series coefficients (complex exponential, trigonometric, and compact forms). In addition, evaluate the signal bandwidth, average value (DC value), and average power.

$$x(t) = \text{rep}_T\left\{A\ \text{rect}\left(t/\tau\right)\right\}$$



Answers. (Notice that $m(t)_{pk} = A$ and *also* $m(t)_{pk-pk} = A$)
Fourier series coefficients:

$$\alpha_n = \frac{A\tau}{T}\text{sinc}\left(\frac{n\omega_o\tau}{2\pi}\right), |\alpha_n| = \frac{A\tau}{T}\left|\text{sinc}\left(\frac{n\omega_o\tau}{2\pi}\right)\right|,\ \angle\alpha_n = 0^\circ\ or\ 180^\circ$$

$$a_n = \frac{2A\tau}{T}\text{sinc}\left(\frac{n\omega_o\tau}{2\pi}\right),\ b_n = 0$$

$$c_n = \frac{2A\tau}{T}\left|\text{sinc}\left(\frac{n\omega_o\tau}{2\pi}\right)\right|,\ \theta_n = 0^\circ\ or\ 180^\circ$$

Bandwidth:

$$B_{x(t)} = 1/\tau$$

DC value:

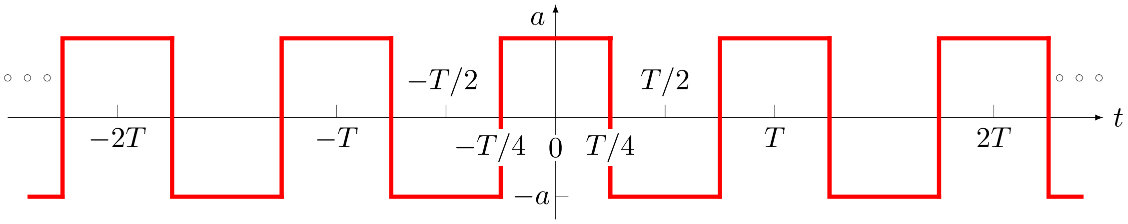
$$\overline{x(t)} = A\tau/T$$

Average Power:

$$P_x = \overline{x^2(t)} = A^2\tau/T$$

Exercise 2. Repeat the above exercise for the signal $x(t)$ shown below.

$$x(t) = \text{rep}_T\left\{2a\ \text{rect}\left(2t/T\right)\right\} - a$$



Answers. (Notice that $m(t)_{pk} = a$, which is *different* than $m(t)_{pk-pk} = 2a$)
Fourier series coefficients:

$$\alpha_0 = 0,\ \ \alpha_n = a\ \text{sinc}\left(\frac{n}{2}\right),$$

$$|\alpha_0| = 0,\ \ |\alpha_n| = a\left|\text{sinc}\left(\frac{n}{2}\right)\right|, \angle\alpha_n = 0^\circ\ \mathit{or}\ 180^\circ$$

$$a_0 = 0,\ \ b_0 = 0, a_n = 2a\ \text{sinc}\left(\frac{n}{2}\right),\ \ b_n = 0$$

$$c_0 = 0, c_n = 2a\left|\text{sinc}\left(\frac{n}{2}\right)\right|,\ \ \theta_n = 0^\circ\ \mathit{or}\ 180^\circ$$

Bandwidth:

$$B_{x(t)} = 2/T$$

DC value:

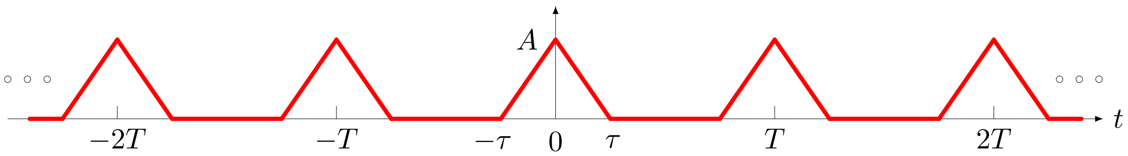
$$\overline{x(t)} = 0$$

Average Power:

$$P_x = \overline{x^2(t)} = 2a^2 - a^2 = a^2$$

Exercise 3. Repeat the above exercise for the signal $x(t)$ shown below.

$$x(t) = \text{rep}_T\left\{A\ \Delta\left(t/\tau\right)\right\}$$



Answers. (Notice that $m(t)_{pk} = A$ and *also* $m(t)_{pk-pk} = A$)
Fourier series coefficients:

$$\alpha_n = \frac{A\tau}{T}\text{sinc}^2\left(\frac{n\omega_o\tau}{2\pi}\right),\ \ |\alpha_n| = \frac{A\tau}{T}\text{sinc}^2\left(\frac{n\omega_o\tau}{2\pi}\right),\ \ \angle\alpha_n = 0^\circ$$

$$a_n = \frac{2A\tau}{T}\text{sinc}^2\left(\frac{n\omega_o\tau}{2\pi}\right),\ \ b_n = 0$$

$$c_n = \frac{2A\tau}{T}\text{sinc}^2\left(\frac{n\omega_o\tau}{2\pi}\right),\ \theta_n = 0^\circ$$

Bandwidth:

$$B_{x(t)} = 1/\tau$$

DC value:

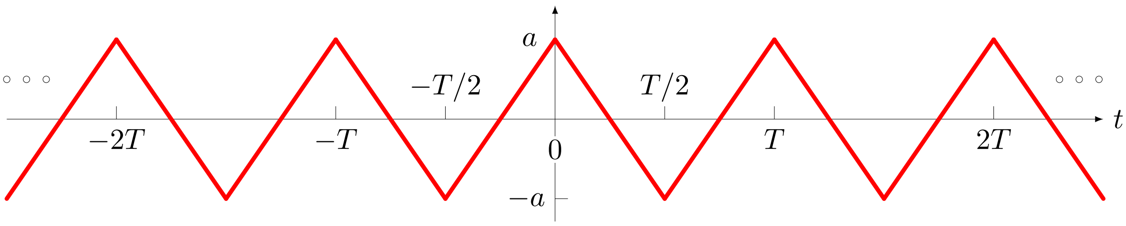
$$\overline{x(t)} = A\tau/T$$

Average Power:

$$P_x = \overline{x^2(t)} = 2A^2\tau/3T$$

Exercise 4. Repeat the above exercise for the signal $x(t)$ shown below.

$$x(t) = \text{rep}_T\left\{2a\ \Delta\left(2t/T\right)\right\} - a$$



Answers. (Notice that $m(t)_{pk} = a$, which is *different* than $m(t)_{pk-pk} = 2a$)
Fourier series coefficients:

$$\alpha_0 = 0,\ \alpha_n = a\ \text{sinc}^2\left(\frac{n}{2}\right)$$

$$|\alpha_0| = 0,\ |\ \alpha_n| = a\ \text{sinc}^2\left(\frac{n}{2}\right),\ \angle\alpha_n = 0^\circ$$

$$a_0 = 0,\ b_0 = 0, a_n = 2a\ \text{sinc}^2\left(\frac{n}{2}\right),\ b_n = 0$$

$$c_0 = 0, c_n = 2a\ \text{sinc}^2\left(\frac{n}{2}\right),\ \theta_n = 0^\circ$$

Bandwidth:

$$B_{x(t)} = 2/T$$

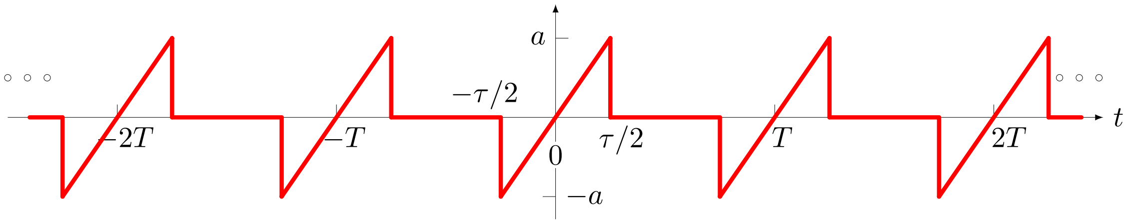
DC value:

$$\overline{x(t)} = 0$$

Average Power:

$$P_x = \overline{x^2(t)} = 4a^2/3 - a^2 = a^2/3$$

Exercise 5. Repeat the above exercise for the signal $x(t)$ shown below.



Answers. (Notice that $m(t)_{pk} = a$, which is *different* than $m(t)_{pk-pk} = 2a$)
Fourier series coefficients:

$$\alpha_0 = 0,\ \alpha_n = \frac{-j2a}{n\omega_o T}\left[\text{sinc}\left(\frac{n\omega_o\tau}{2\pi}\right) - \cos\left(\frac{n\omega_o\tau}{2}\right)\right],$$

$$|\alpha_0| = 0,\ |\alpha_n| = \frac{2a}{n\omega_o T}\left|\text{sinc}\left(\frac{n\omega_o\tau}{2\pi}\right) - \cos\left(\frac{n\omega_o\tau}{2}\right)\right|,\ \angle\alpha_n = \pm 90^\circ$$

$$a_0 = 0,\ b_0 = 0,\ a_n = 0,\ b_n = \frac{4a}{n\omega_o T}\left[\text{sinc}\left(\frac{n\omega_o\tau}{2\pi}\right) - \cos\left(\frac{n\omega_o\tau}{2}\right)\right]$$

$$c_0 = 0,\ c_n = \frac{4a}{n\omega_o T}\left|\text{sinc}\left(\frac{n\omega_o\tau}{2\pi}\right) - \cos\left(\frac{n\omega_o\tau}{2}\right)\right|,\ \theta_n = \mp 90^\circ$$

Bandwidth:

$$B_{x(t)} = 3/\tau$$

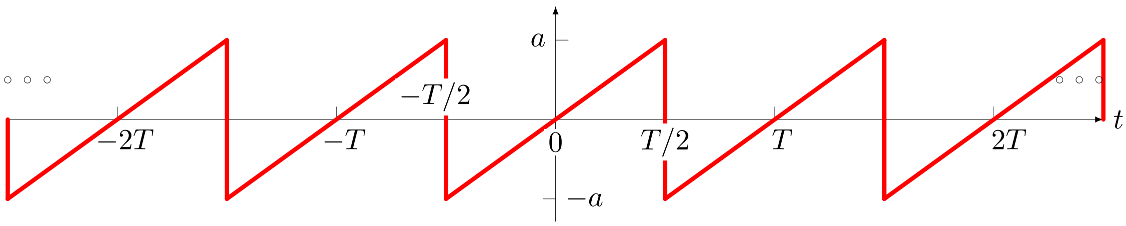
DC value:

$$\overline{x(t)} = 0$$

Average Power:

$$P_x = \overline{x^2(t)} = a^2\tau/3T$$

Exercise 6. Repeat the above exercise for the signal $x(t)$ shown below.



Answers. (Notice that $m(t)_{pk} = a$, which is *different* than $m(t)_{pk-pk} = 2a$)
Fourier series coefficients:

$$\alpha_0 = 0, \alpha_n = \frac{ja\cos(n\pi)}{n\pi}$$

$$|\alpha_0| = 0, \; |\alpha_n| = \left|\frac{a\cos(n\pi)}{n\pi}\right|, \; \angle\alpha_n = \pm 90^\circ$$

$$a_0 = 0, \; b_0 = 0, \; a_n = 0, \; b_n = \frac{-2a\cos(n\pi)}{n\pi}$$

$$c_0 = 0, \; c_n = \left|\frac{2a\cos(n\pi)}{n\pi}\right|, \; \theta_n = \mp 90^\circ$$

Bandwidth:

$$B_{x(t)} = 3/T$$

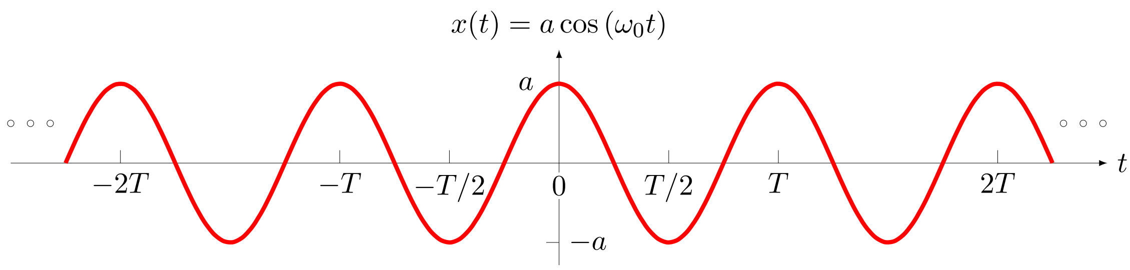
DC value:

$$\overline{x(t)} = 0$$

Average Power:

$$P_x = \overline{x^2(t)} = a^2/3$$

Exercise 7. Repeat the above exercise for the signal $x(t)$ shown below.



Answers. (Notice that $m(t)_{pk} = a$, which is *different* than $m(t)_{pk-pk} = 2a$)
Fourier series coefficients:

$$\alpha_1 = \frac{a}{2}, \alpha_{-1} = \frac{a}{2}, \alpha_n = 0, \; n \neq \pm 1$$

$$|\alpha_1| = \frac{a}{2}, \; \angle\alpha_1 = 0^\circ, |\alpha_{-1}| = \frac{a}{2}, \; \angle\alpha_{-1} = 0^\circ, |\alpha_n| = 0, \; \angle\alpha_n = 0^\circ, \; n \neq 1$$

$$a_0 = 0, \; b_n = 0, a_1 = a, \; a_n = 0, \; n \neq 1$$

$$c_0 = 0, c_1 = a, \; \theta_1 = 0^\circ, \; c_n = 0, \; \theta_n = 0^\circ, \; n \neq 1$$

Bandwidth:

$$B_{x(t)} = 1/T$$

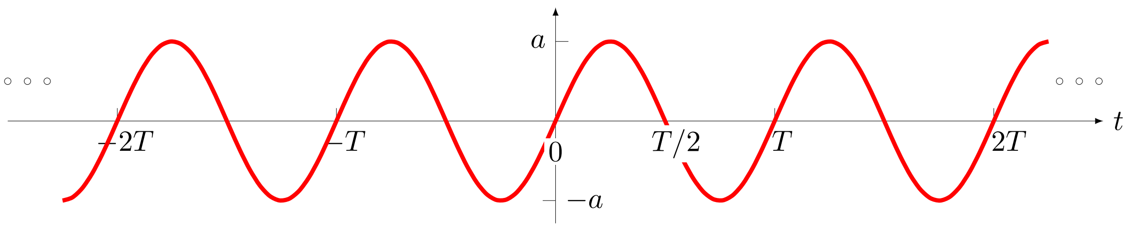
DC value:

$$\overline{x(t)} = 0$$

Average Power:

$$P_x = \overline{x^2(t)} = a^2/2$$

Exercise 8. Repeat the above exercise for the signal $x(t)$ shown below.



Answers. (Notice that $m(t)_{pk} = a$, which is *different* than $m(t)_{pk-pk} = 2a$)
Fourier series coefficients:

$$\alpha_1 = -j\frac{a}{2}, \alpha_{-1} = j\frac{a}{2}, \alpha_n = 0,\ n \neq \pm 1$$

$$|\alpha_1| = \frac{a}{2},\ \angle\alpha_1 = -90^\circ, |\alpha_{-1}| = \frac{a}{2},\ \angle\alpha_{-1} = 90^\circ, |\alpha_n| = 0,\ \angle\alpha_n = 0^\circ,\ n \neq 1$$

$$a_0 = 0,\ a_n = 0, b_1 = a,\ b_n = 0,\ n \neq 1$$

$$c_0 = 0, c_1 = a,\ \theta_1 = 90^\circ,\ c_n = 0,\ \theta_n = 0^\circ,\ n \neq 1$$

Bandwidth:

$$B_{x(t)} = 1/T$$

DC value:

$$\overline{x(t)} = 0$$

Average Power:

$$P_x = \overline{x^2(t)} = a^2/2$$

Notice that the fundamental frequency of any of the above periodic signals is $\omega_0 = \frac{2\pi}{T}$ rad/s or $f_0 = \frac{1}{T}$ Hz, and the *fundamental frequency* f_0 should not be confused with the *bandwidth* $B_{x(t)}$ of the signal.

Please note that the **complex** Fourier series coefficients α_n's represent the **Fourier spectrum** of the signal $x(t)$, or simply, its **spectrum**. Since these coefficients are complex numbers, they generate two spectra: the **magnitude spectrum** $|\alpha_n|$ and the **phase spectrum** $\angle\alpha_n$ of $x(t)$. The magnitude spectrum can also be drawn using the compact coefficients c_n since $c_n = 2|\alpha_n|$. The c_n spectrum, however, is called the **one-sided magnitude spectrum** (because $n \geq 0$), while the $|\alpha_n|$ spectrum is called the **two-sided magnitude spectrum** (because $-\infty < n < +\infty$).

3 Fourier Transform

The Fourier transform is a mathematical tool that converts a signal $x(t)$ from *time domain* into *frequency domain* as $X(\omega)$. The inverse Fourier transform does the opposite.

3.1 Definition

The Fourier transform of a general signal $x(t)$, whether *periodic* or *aperiodic*, is given by:

$$X(\omega) = \mathscr{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

and the inverse Fourier transform is:

$$x(t) = \mathscr{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

The Fourier transform $X(\omega)$ represents the **Fourier spectrum density** of the signal $x(t)$. Notice that the Fourier spectrum density of *periodic* signals consists of a group of impulses $\delta(\omega)$'s, while the Fourier spectrum density of *aperiodic* signals is a smooth continuous curve. This is due to the fact that periodic signals are actually the sum of an infinite number of sinusoidals.

3.2 Properties of Fourier Transform

When evaluating Fourier transform, we typically avoid using the original integral and rely on using tables instead. See Table 1 (Selected Fourier Transform Pairs) and Table 2 (Properties of Fourier Transform). Make sure you memorize both tables.

Notice that all Fourier transforms in Table 1 are given in terms of angular frequency ω (rad/s) instead of ordinary frequency f (Hz). This is the convention we will use in this class. There is a factor of 2π that you have to be aware of between the Fourier transform $X(\omega)$ and $X(f)$. For example, $\mathscr{F}\{x(t) = a\cos(\omega_0 t)\} = X(\omega) = \pi a\delta(\omega - \omega_0) + \pi a\delta(\omega + \omega_0)$. However, $\mathscr{F}\{x(t) = a\cos(\omega_0 t)\} = X(f) = \frac{a}{2}\delta(f - f_0) + \frac{a}{2}\delta(f + f_0)$. Similarly, $\mathscr{F}\{x(t) = \mathrm{rect}(t)\} = X(\omega) = \mathrm{sinc}\left(\frac{\omega}{2\pi}\right)$, but $\mathscr{F}\{x(t) = \mathrm{rect}(t)\} = X(f) = \mathrm{sinc}(f)$. This difference is due to the fact that $\omega = 2\pi f$.

$x(t)$	$X(\omega) = \mathscr{F}\{x(t)\}$
$a\cos(\omega_0 t)$	$\pi a\delta(\omega-\omega_o) + \pi a\delta(\omega+\omega_o)$
$a\sin(\omega_0 t)$	$-j\pi a\delta(\omega-\omega_o) + j\pi a\delta(\omega+\omega_o)$
$e^{\pm j\omega_0 t}$	$2\pi\delta(\omega \mp \omega_0)$
$\text{rect}\,(t)$	$\text{sinc}\left(\frac{\omega}{2\pi}\right)$
$\text{rect}\left(\frac{t}{\tau}\right)$	$\tau\,\text{sinc}\left(\frac{\omega\tau}{2\pi}\right)$
$\Delta\,(t)$	$\text{sinc}^2\left(\frac{\omega}{2\pi}\right)$
$\Delta\left(\frac{t}{\tau}\right)$	$\tau\,\text{sinc}^2\left(\frac{\omega\tau}{2\pi}\right)$
$\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$	$\text{rect}\left(\frac{\omega}{2\pi}\right)$
$\text{sinc}\left(\frac{t}{2\pi}\right)$	$2\pi\ \text{rect}\,(\omega)$
$\text{saw}\left(\frac{t}{\tau}\right)$	$\frac{-2j}{\omega}\left[\text{sinc}\left(\frac{\omega\tau}{2\pi}\right) - \cos\left(\frac{\omega\tau}{2}\right)\right]$
$\delta(t)$, Dirac delta function	1
1	$2\pi\delta(\omega)$
$\text{rep}_T\{p(t)\}$, periodic	$\sum_{n=-\infty}^{\infty} 2\pi\alpha_n\delta(\omega-n\omega_o)$
$\text{rep}_T\{\delta(t)\} = \sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\sum_{n=-\infty}^{\infty}\frac{2\pi}{T}\delta(\omega-n\omega_o) = \omega_0\text{rep}_{\omega_0}\{\delta(\omega)\}$
$\text{rep}_T\left\{A\ \text{rect}\left(\frac{t}{\tau}\right)\right\}$	$\sum_{n=-\infty}^{\infty} 2\pi\frac{A\tau}{T}\text{sinc}\left(\frac{n\omega_o\tau}{2\pi}\right)\delta(\omega-n\omega_o)$
$\text{rep}_T\left\{A\ \Delta\left(\frac{t}{\tau}\right)\right\}$	$\sum_{n=-\infty}^{\infty} 2\pi\frac{A\tau}{T}\text{sinc}^2\left(\frac{n\omega_o\tau}{2\pi}\right)\delta(\omega-n\omega_o)$
$\text{rep}_T\left\{a\ \text{saw}\left(\frac{t}{T}\right)\right\}$	$\sum_{n=-\infty}^{\infty} j2\pi a\frac{\cos(n\pi)}{n\pi}\delta(\omega-n\omega_o)$
$u(t)$, unit step function	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\text{sgn}(t) = u(t) - u(-t)$	$\frac{2}{j\omega}$
$e^{-t^2/(2\sigma^2)}$	$\sigma\sqrt{2\pi}\ e^{-\sigma^2\omega^2/2}$
$e^{-a\|t\|},\quad a>0$	$\frac{2a}{a^2+\omega^2}$

Property	$x(t)$	$X(\omega) = \mathscr{F}\{x(t)\}$
Linearity (superposition)	$ax(t) + by(t)$	$aX(\omega) + bY(\omega)$
Complex conjugate	$x^*(t)$	$X^*(-\omega)$
Symmetry	$x_{even}(t)$ $x_{odd}(t)$	$X_{even}(\omega)$, real $X_{odd}(\omega)$, imaginary
Duality	$X(t)$	$2\pi x(-\omega)$
Time scaling (reciprocal spreading)	$x\left(\frac{t}{\tau}\right)$	$\|\tau\|\ X(\tau\omega)$
Time inversion (time reversal)	$x(-t)$	$X(-\omega)$
Time shift (time delay/advance)	$x(t \pm t_0)$	$X(\omega)e^{\pm j\omega t_0}$
Frequency shift	$x(t)e^{\pm j\omega_0 t}$	$X(\omega \mp \omega_0)$
Modulation	$x(t)\cos(\omega_0 t) =$ $\frac{x(t)}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$	$\frac{1}{2}X(\omega-\omega_0) + \frac{1}{2}X(\omega+\omega_0)$
Time differentiation	$\frac{d^n}{dt^n}x(t)$	$(j\omega)^n\ X(\omega)$
Time integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{X(\omega)}{j\omega} + \pi X(0)\ \delta(\omega)$
Time convolution	$x(t) \circledast y(t)$	$X(\omega)Y(\omega)$
Frequency convolution	$x(t)y(t)$	$\frac{1}{2\pi}\left(X(\omega) \circledast Y(\omega)\right)$

4 Energy and Power Spectral Densities

4.1 Energy Spectral Density

The **energy spectral density** (ESD) of a general signal $x(t)$ is defined as:

$$\text{ESD} = \Psi_x(\omega) = |X(\omega)|^2$$

The ESD is a function that describes the relative amount of energy of a given signal versus frequency. The total area under the ESD is the total energy in the signal $x(t)$, denoted by E_x.

4.2 Parseval's Theorem

The total energy in a general signal $x(t)$ can be calculated either from time-domain or frequency-domain as follows:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

4.3 Power Spectral Density

The **power spectral density** (PSD) of a general signal $x(t)$ is defined as:

$$\text{PSD} = S_x(\omega) = \lim_{T \to \infty} \frac{1}{T} |X_T(\omega)|^2$$

The PSD is a function that describes the relative amount of power of a given signal versus frequency. The total area under the PSD is the average power in the signal $x(t)$, denoted by P_x.

Notice that the PSD of *periodic* functions consists of a group of impulses $\delta(\omega)$'s, while the PSD of *aperiodic* functions is a smooth continuous curve. This is due to the fact that periodic functions are actually the sum of an infinite number of sinusoidal functions.

Most of the signals we consider in communications theory exist for a long time, i.e., they are power signals. Some of them are periodic and some are aperiodic. Power signals have a PSD, not an ESD (their ESD is infinite).

4.4 Average Power

The average power in a general signal $x(t)$ can be calculated either from time-domain or frequency-domain as follows:

$$P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega) d\omega$$

5 Average and RMS versus Average Power

Consider a continuous signal $x(t)$ and a discrete version of such signal x_n with a sampling period Δt (the sampling frequency is given by $f_s = \frac{1}{\Delta t}$).

5.1 Average Value

The average value (*or* DC value *or* DC offset *or* DC shift) of the signal $x(t)$ is given by:

$$\text{DC} = \overline{x(t)} = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)dt = \alpha_0$$

and for the sampled version of $x(t)$, where we have N samples Δt apart, the average value is:

$$\overline{x_n} = \frac{1}{N\Delta t} \sum_{n=1}^{N} x_n \Delta t = \frac{1}{N} \sum_{n=1}^{N} x_n, \qquad \Delta t = \frac{1}{f_s}$$

5.2 RMS Value

The *root mean square* (rms) value of the signal $x(t)$ is given by:

$$x_{rms} = \sqrt{\lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t)dt}$$

and for the sampled version of $x(t)$, where we have N samples Δt apart, the rms value is:

$$x_{rms} = \sqrt{\frac{1}{N\Delta t} \sum_{n=1}^{N} x_n^2 \Delta t} = \sqrt{\frac{1}{N} \sum_{n=1}^{N} x_n^2}, \qquad \Delta t = \frac{1}{f_s}$$

5.3 Average Power

As explained earlier, the *average power* in the signal $x(t)$ can be calculated using:

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 \, dt$$

and for the sampled version, x_n, the average power is:

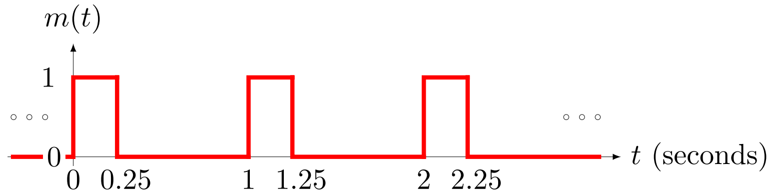
$$P_x = \frac{1}{N} \sum_{n=1}^{N} x_n^2$$

Notice that the square of the rms value is actually the average power in the signal. This is because we are assuming a normalized load impedance of 1 Ω. Hence, the average power in $x(t)$ is:

$$P_x = \frac{x_{rms}^2}{R} = \frac{x_{rms}^2}{1} = x_{rms}^2$$

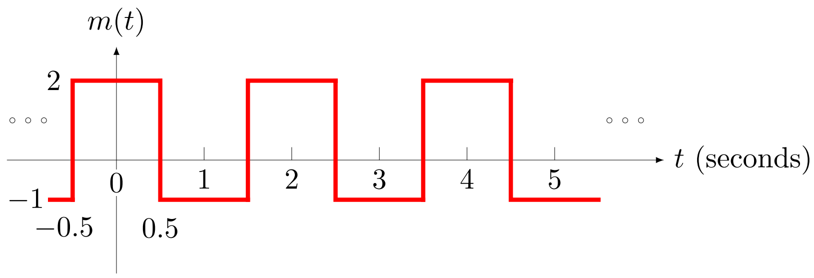
6 Practice

Problem 1. For the signal $m(t)$ shown below, evaluate and sketch the Fourier transform, then evaluate the signal bandwidth, average value (DC value) and average power.



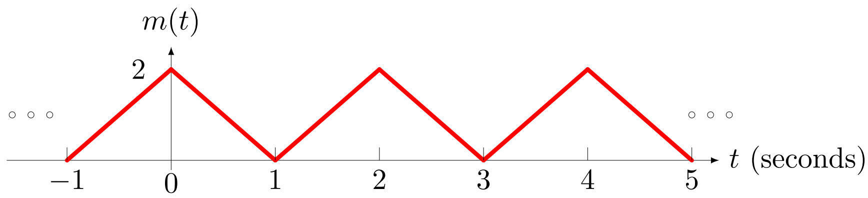
Hint. You can solve this problem by calculating integrals from scratch, but this is *not recommended* since this is time consuming. Rather, use Fourier transform properties, and notice that this signal is a time shifted version of the signal in Exercise 1. You just need to know the effect of time shift on each of the answers.

Problem 2. For the signal $m(t)$ shown below, evaluate and sketch the Fourier transform, then evaluate the signal bandwidth, average value (DC value) and average power.



Hint. Use Fourier transform properties, and notice that this is a DC shifted version of the signal in Exercise 2. You just need to know how a DC shift affects each of the answers. You can use orthogonality to help you evaluate average power.

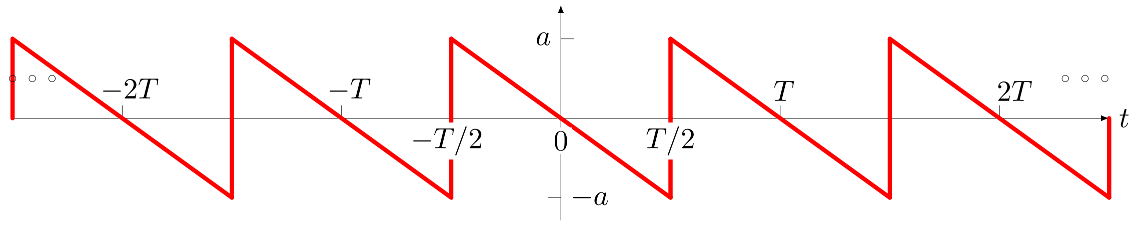
Problem 3. For the signal $m(t)$ shown below, evaluate and sketch the Fourier transform, then evaluate the signal bandwidth, average value (DC value) and average power.



Hint. Use Fourier transform properties. This signal can be thought of as a special case of the signal in Exercise 3, where τ is set to $T/2$. Alternatively, the signal $m(t)$ can be thought of as a DC shifted version of the signal in Exercise 4. In the latter case, direct use of orthogonality is allowed for power calculations.

Problem 4. For the signal $m(t)$ shown below, evaluate and sketch the Fourier transform, then evaluate the signal bandwidth, average value (DC value) and average power.

$$x(t) = -\text{rep}_T\{a \text{ saw}(t/T)\}$$



Hint. This is a vertically inverted version of the signal in Exercise 6. Inversion is simply multiplying the signal by -1. Use this fact and Fourier transform properties to quickly evaluate the answers.